Optimum Quarter-Wave Transformers*

LEO YOUNG†, SENIOR MEMBER, IRE

Summary—**The design of uniformly dispersive quarter-wave transformers is a well explored subject. Common examples are rectangular waveguide E-plane transformers, in which the *a* dimension is kept constant.**

In this paper, it is shown that the performance of conventional quarter-wave transformers of a single section can always be improved by making the middle section less dispersive than the input and output waveguides, and a formula for the optimum *a* dimension is given.

The theory was verified experimentally. In this instance, the improved transformer measured 50 per cent more bandwidth than did the conventional one, and was 25 per cent shorter besides.

INTRODUCTION

IN the design of quarter-wave transformers, it has hitherto always been assumed that the guide wavelength is independent of position along the line. This is so, for instance, for TEM modes, or for TE_{0n} modes in rectangular waveguide where the wide or a dimension is kept constant. Such transformers, having guide wavelength independent of position, are called *homogeneous* transformers.[1] When the guide wavelength varies along the length of the transformer, it is called *inhomogeneous*. The first exact design formulas for ideal homogeneous quarter-wave transformers[2] were given by Collin,[3] who considered up to four sections. The first complete synthesis procedure was given by Riblet.[4] The author later computed extensive numerical tables,[5] which have been checked out experimentally on numerous occasions.

This paper is concerned only with single-section quarter-wave transformers. In particular, it will be shown that the performance of the conventional homogeneous waveguide transformer of a single section can always be improved by making the transformer section less dispersive than the input and output waveguides, and that an optimum inhomogeneous transformer exists in general. Transformers with two or more sections[6] are not considered in this paper.

THE INSERTION LOSS FUNCTION, P_L

Consider a single-section rectangular waveguide transformer operating in the TE_{01} mode. Let a denote the wide dimension, and b the height, of waveguide. The input guide has dimensions $a_0 \times b_0$, the output guide $a_2 \times b_2$, and the quarter-wave section $a_1 \times b_1$, as shown in Fig. 1.

The transformer is shown schematically in Fig. 2. The characteristic impedances are Z_0, Z_1 and Z_2. The reflection and transmission coefficients[6,7] at the two transformer steps, are Γ_1, Γ_2 and T_1, T_2 given by

$$\Gamma_1 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \qquad \Gamma_2 = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{1}$$

and

$$T_1 = \frac{2(Z_1 Z_0)^{1/2}}{Z_1 + Z_0}, \qquad T_2 = \frac{2(Z_2 Z_1)^{1/2}}{Z_2 + Z_1}. \tag{2}$$

The over-all transfer or wave matrix[6,7] of the transformer can then be written

$$\boldsymbol{T} = \frac{1}{T_1 T_2}\begin{pmatrix} 1 & \Gamma_1 \\ \Gamma_1 & 1 \end{pmatrix}\begin{pmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{pmatrix}\begin{pmatrix} 1 & \Gamma_2 \\ \Gamma_2 & 1 \end{pmatrix} \tag{3}$$

where θ is the electrical length of the transformer section at any frequency. If we write $T = (T_{ij})$, $i, j = 1, 2$, the insertion loss function P_L is

$$\begin{aligned} P_L &= |T_{11}|^2 \\ &= \frac{1}{T_1^2 T_2^2} \left| e^{j\theta} + \Gamma_1 \Gamma_2 e^{-j\theta} \right|^2, \end{aligned} \tag{4}$$

which after manipulation reduces to

$$P_L = 1 + \frac{1}{T_1^2 T_2^2}\left[(\Gamma_2 - \Gamma_1)^2 + 4\Gamma_1\Gamma_2 \cos^2 \theta\right]. \tag{5}$$

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† Stanford Res. Inst., Menlo Park, Calif. Formerly with Electronics Div., Westinghouse Electric Corp., Baltimore, Md.

[1] L. Young, "Concerning Riblet's theorems," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-7, pp. 477–478; October, 1959.

[2] The junction of the two transmission lines when junction discontinuities are neglected is called an "ideal transformer." (This is analogous to two perfectly-coupled coils of turns ratio $(Z_2/Z_1)^{1/2}$ and having infinite inductance.)

[3] R. E. Collin, "Theory and design of wide-band multisection quarter-wave transformers," PROC. IRE, vol. 43, pp. 179–185; February, 1955.

[4] H. J. Riblet, "General synthesis of quarter-wave impedance transformers," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. 5, pp. 36–43; January, 1957.

[5] L. Young, "Tables for cascaded homogeneous quarter-wave transformers," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-7, pp. 233–237; April, 1959.

[6] L. Young, "Design of Microwave Stepped Transformers with Applications to Filters," Ph.D. dissertation, the Johns Hopkins University, Baltimore, Md.; April, 1959.

[7] G. L. Ragan, "Microwave Transmission Circuits," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 9; 1951.

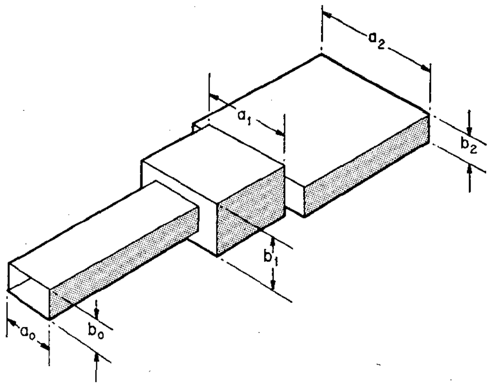

Fig. 1—Quarter-wave transformer (showing notation).

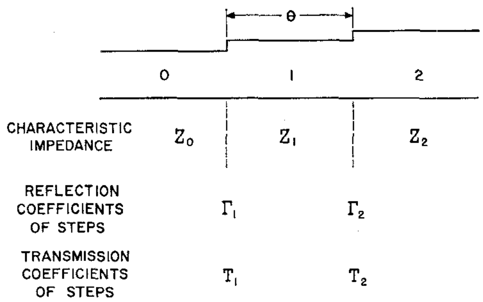

Fig. 2—Quarter-wave transformer parameters.

For a quarter-wave transformer, a match is required at $\theta=\pi/2$, so

$$\Gamma_2 = \Gamma_1 \quad \text{at} \quad \theta = \frac{\pi}{2}\cdot \tag{6}$$

Some Formulas for Waveguides

Denote the free space (or medium) wavelength by λ and define the differential operator

$$D = \lambda \frac{d}{d\lambda} = \frac{d}{d(\ln C\lambda)} \tag{7}$$

where C is any constant. Define the dimensionless ratios

$$s = \frac{\lambda_g}{\lambda_c} \tag{8}$$

$$t = \frac{\lambda_g}{\lambda} \tag{9}$$

$$u = \frac{\lambda}{\lambda_c} \tag{10}$$

where λ_g is the guide wavelength and λ_c is the cutoff wavelength of the waveguide, for the mode of propagation.

Since

$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda^2} - \frac{1}{\lambda_c^2}, \tag{11}$$

then

$$t^2 - s^2 = 1. \tag{12}$$

Also

$$ut = s. \tag{13}$$

Their derivatives are

$$Dt = s^2 t \tag{14}$$

$$Ds = st^2 \tag{15}$$

$$Du = u. \tag{16}$$

The derivative of the electrical length θ is given by

$$D\theta = -t_1^2\theta. \tag{17}$$

For TE modes,

$$Z_{(TE\text{ mode})} \propto \frac{b}{a}\cdot\frac{\lambda_g}{\lambda}, \tag{18}$$

and remembering that a and b are constants for the differential operator $D=\lambda(d/d\lambda)$, it can be shown[6] that

$$D\Gamma_1 = \tfrac{1}{2}(1-\Gamma_1^2)(t_1^2 - t_0^2)$$

$$D\Gamma_2 = \tfrac{1}{2}(1-\Gamma_2^2)(t_2^2 - t_1^2). \tag{19}$$

The Optimum Transformer

For a quarter-wave single-section transformer, (6) ensures a perfect match at one frequency, the "center frequency." This equation does not, however, completely determine the transformer if dispersive, since the rates of change of Γ_1, Γ_2 and θ may still be adjusted by one remaining parameter, the cutoff wavelength of the intermediate section. For optimum performance set

$$D^2 P_L = 0 \quad \text{at} \quad \theta = \frac{\pi}{2}\cdot \tag{20}$$

From (5), this becomes

$$D(\Gamma_2 - \Gamma_1)^2 + 4\Gamma^2(D\theta)^2 = 0 \tag{21}$$

where use has been made of (6) after differentiation. Referring to (17) and (19) one finally obtains

$$t_1^2 = \frac{1}{2}\left[\frac{t_0^2 + t_2^2}{1 + \left(\dfrac{\pi\Gamma}{1-\Gamma^2}\right)^2}\right] \tag{22}$$

as the required condition, where $\Gamma_1=\Gamma_2=\Gamma$ by (6). This can also be expressed

$$\lambda_{g_1}^2 = \frac{1}{2}\left[\frac{\lambda_{g_0}^2 + \lambda_{g_2}^2}{1 + \left(\dfrac{\pi}{4}\right)^2 \dfrac{(Z_2 - Z_0)^2}{Z_2 Z_0}}\right]. \tag{23}$$

Note that

$$\lambda_{g_1}^2 \leq \tfrac{1}{2}(\lambda_{g_0}^2 + \lambda_{g_2}^2) \tag{24}$$

(equal only if $\Gamma = 0$); and if furthermore $a_0 = a_2$, then

$$a_{1\,\text{optimum}} > a_0 = a_2 \tag{25}$$

in all cases, *i.e.*, in this case the matching section should always be less dispersive than the input and output waveguides (as might have been expected). Thus, a homogeneous transformer is never an optimum transformer (except in the trivial case $\Gamma = 0$). A flatter frequency response can be obtained, at least for small bandwidths, by making the transformer inhomogeneous.

In general, if Γ is large enough, or t_0 and t_2 small enough, or both, (22) may yield a value for t_1 less than unity, and a true optimum transformer then does not exist. In that case, $t_1 = 1$ gives the best transformer.

Numerical Results

To test the theory and to assess the sharpness of the optimum, we analyzed numerically several transformers. In all cases, the optimum transformer was correctly predicted by (22).

The numerical work shows that the improvement obtained in going from a homogeneous to an inhomogeneous transformer of one section is significant only for fairly large transformer ratios and more-than-average dispersive lines. One such example is reproduced below.

It is required to transform from 0.900 inch×0.050 inch to 0.900 inch×0.400 inch waveguide at a design wavelength of $\lambda_0 = 1.638$ inches. The intermediate section dimensions will be denoted by $a_1 \times b_1$. The VSWR against wavelength response of the homogeneous ($a_1 = 0.900$ inch) and optimum ($a_1 = 1.90$ inches) transformers are plotted in Fig. 3. The slope at the center of the optimum transformer curve is less than half (about 45 per cent) of the corresponding slope for the homogeneous transformer. Again, further computations show that the response curve changes slowly near the optimum, and most of the improvement occurs as one pulls away from the homogeneous case. Thus, more than half the improvement is realized in changing a_1 from 0.900 to 0.990 inch, an increase in width of only ten per cent. This response is also plotted in Fig. 3. Eq. (22) has also been verified numerically where the input and output waveguide wide dimensions ("a-dimensions") are different, and therefore a homogeneous transformer is not possible at all.

Practical Considerations for Coaxial Line and TE_{01} Mode Rectangular Waveguide Transformers

All exact synthesis procedures assume the existence of ideal transformers. These produce only a change in characteristic impedance, which occurs in the plane of the transformer.

The departure from ideal conditions can be explained in more than one way. Thus, any obstacle can be represented as a transformer *at one frequency* by choosing

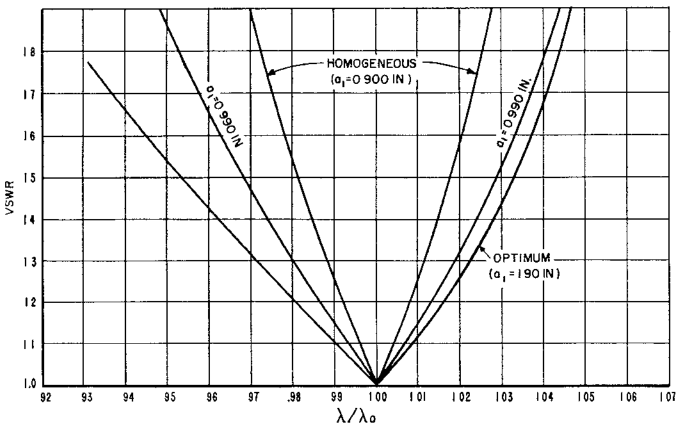

Fig. 3—VSWR against wavelength of homogeneous and optimum inhomogeneous transformers, as well as one intermediate transformer.

reference planes with real Γ. However, as the frequency changes,

1) the magnitude of the junction VSWR changes differently from the impedance ratio of the two waveguides,
2) the left reference plane moves,
3) the right reference plane moves.

Thus, the frequency behavior of *three parameters* must be given to describe completely a lossless two-port.

Coaxial Line and Waveguide E-Plane Steps

Coaxial line junctions and rectangular waveguides with E-plane steps can be represented by an ideal transformer plus a shunt susceptance at the discontinuity.[8,9] Formulas and graphs are given by Marcuvitz.[10] For this type of junction, the transformer ratio is also independent of frequency, and equal to the impedance ratio. The susceptance value is positive, *i.e.* capacitive, but its frequency dependence is different from that of a capacity. Since the discontinuity susceptance is usually small, its effect on the amplitude of the (real) reflection coefficient is second-order, and the only first-order manifestation is a phase-shift through the transformer. It can, therefore, be compensated quite accurately by changes in length, usually a decrease from the quarter-wave spacings between the steps.[11]

Waveguide H-Plane Steps

When a change in the wide or a dimension occurs in rectangular waveguide propagating in the TE_{01} mode, the discontinuity may be represented[8,9] by an ideal transformer modified by a shunt susceptance which is negative (inductive). However, this alone is not suf-

[8] N. Marcuvitz, "Waveguide Handbook," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 10; 1951.
[9] L. Lewin, "Advanced Theory of Waveguides," Iliffe and Sons, Ltd., London, Eng.; 1951.
[10] Marcuvitz, *op. cit.*, pp. 307–312.
[11] S. B. Cohn, "Optimum design of stepped transmission-line transformers," IRE Trans. on Microwave Theory ahd Techniques, vol. MTT-3, pp. 16–21; April, 1955.

ficient. Unlike for the E-plane step, the maximum number of three parameters must be specified. In Marcuvitz,[12] the three parameters chosen and graphed are

1) a shunt inductance;
2) the transformer impedance ratio;
3) the position of one reference plane; the position of the other reference plane is defined to be in the plane of the junction.

The transformer impedance ratio of an H-plane step is no longer simply the ratio of the characteristic impedances of the two waveguides, as is the case for E-plane steps, but is greater. Thus, even after the inductances are compensated for by a change in the spacings (as for the capacitances at E-plane steps), there remains an increase in the transformer ratio of the H-plane step over and above the impedance ratio of the two waveguides.

The practical effects on the design of H-plane transformers are usually as follows:

1) The effect of the shunt inductance can be corrected for by Cohn's method.[11]
2) The increase in transformer ratios has the effect of increasing the effective value of the output-to-input impedance ratio R. The effective R is the product of all the transformer ratios. One also has to compensate the individual section characteristic impedances, which can be done by simply adjusting the waveguide heights.
3) The distance of the reference plane from the junction increases from zero approximately as the square of (a_1-a_2), where a_1 and a_2 are the two guide widths ("a-dimensions"). This distance can become considerably larger than a quarter-wave length for H-plane steps in excess of 10 or 20 per cent of the guide-widths. In practice, a first-order correction might therefore be expected to hold only for steps smaller than this. In general, as cutoff is approached, the ratio (λ_g/a) becomes large, and the correction will become a smaller fraction of $\frac{1}{4}\lambda_g$, and so proportionately larger steps might then be corrected for.

Compound Steps in Waveguide

For compound steps in both the E-plane and the H-plane simultaneously (changes in both a and b dimensions at one junction of two guides), no formulas or numerical information are available. If the necessary corrections are small enough, it should be possible to treat the E-plane capacity correction, and the two H-plane corrections for inductance and reference plane position, separately. Then add the three corrections to each section length as if they were independent.

Finally, symmetrical steps generally require less compensation than asymmetrical steps, and are for this reason to be preferred.

[12] Marcuvitz, *op. cit.*, pp. 292–304.

Experimental Verification

Two of the transformers described in the numerical example, the homogeneous (conventional) transformer with $a_1=0.900$ inch and the inhomogeneous (improved) transformer with $a_1=0.990$ inch, were built and tested. (The optimum transformer, with $a_1=1.90$ inches would have introduced higher-order modes, besides performing theoretically little better than the transformer with $a_1=0.990$ inch, and was therefore not constructed. Compare Fig. 3.) The free space wavelength of 1.638 inches corresponds to a frequency of 7211 mc per second. The output waveguide size of 0.900 inch$\times$0.400 inch is Retma waveguide type No. WR90 and could conveniently be connected directly to a standard X-band slotted line. For the output waveguide of cross section 0.900 inch$\times$0.050 inch, a special sliding load was constructed which had a VSWR of better than 1.02 over most of the frequency band covered in the tests. The inside dimensions of the two transformers, including the intermediate sections after correction for the junction susceptances and reference plane positions[8,10] are shown in Fig. 4. All steps were symmetrical.

The VSWR against wavelength response of these two transformers (Fig. 4), treated as ideal transformers, corresponds to the upper two curves in Fig. 3. The inhomogeneous response curve (middle curve in Fig. 3) has to be modified to allow for the junctions not being ideal, which increases the effective transformer ratio, R, of the inhomogeneous transformer by some factor. This factor is determined from a figure in Marcuvitz' work,[13] for the symmetrical step, and in this case turns out to be about 1.095 per step; hence, the effective R equals $(1.095)^2=1.20$ times 8, or 9.6.

Thus, after making the appropriate length corrections, the electrical performance of this transformer [Fig. 4(b)] should correspond to the following ideal transformer:

Input waveguide:	0.900 inch$\times$0.050 inch
Middle section:	0.990 inch$\times$0.231 inch
Output waveguide:	0.900 inch$\times$0.480 inch

at a frequency of 7211 Mc.

The VSWR against wavelength response of this ideal transformer is shown as curve (c) in Fig. 5. This, therefore, becomes the expected performance of the improved transformer [Fig. 4(b)], whose ideal response corresponds to the middle curve in Fig. 3, or curve (b) in Fig. 5. Finally, curve (a) in Fig. 5 is both the ideal and the expected performance of the *homogeneous* transformer [Fig. 4(a)].

The measured points for the two transformers shown in Fig. 4 are plotted in Fig. 6, together with their expected (computed) curves. It is seen that the experimental points for both the conventional (homogeneous) and improved (inhomogeneous) transformers lie close

[13] Marcuvitz, *op. cit.*, p. 299, Fig. 5.24-2.

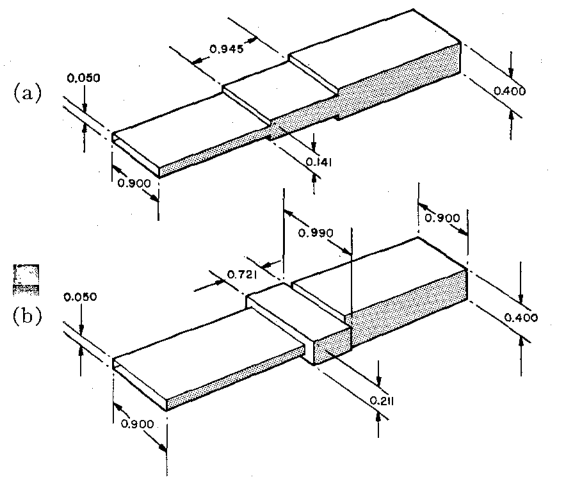

Fig. 4—One-section transformers: (a) control transformer (homogeneous); (b) improved transformer (inhomogeneous).

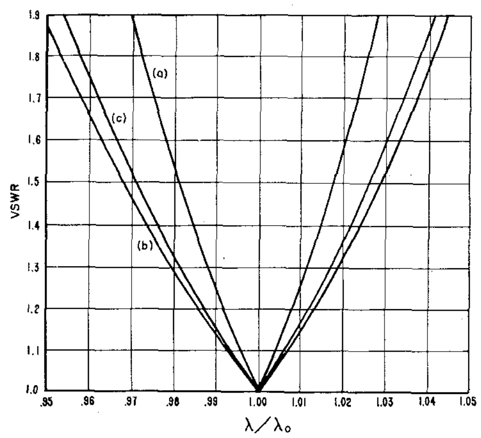

Fig. 5—VSWR against wavelength of three transformers of one section: (a) homogeneous transformer; (b) inhomogeneous transformer without allowing for transformer ratio increase; (c) same inhomogeneous transformer assuming increase of transformer ratio as determined from Marcuvitz.[8]

to the computed curves, and bear out the theory to within experimental accuracy. (The spread in the measured points is thought to be due mainly to the presence of harmonic frequencies from the signal generator.)

Conclusion

We conclude, therefore, that the ideal transformer theory applies to inhomogeneous as well as homogeneous transformers in rectangular waveguide after computable corrections[8,10] have been applied. In particular, a single-section inhomogeneous transformer is capable of bettering the performance of the homogeneous one, and this improvement is obtained with a shorter transformer.

It is interesting to note that Solymar[14] and Riblet[15] have recently compared quarter-wave transformers to other impedance transformers and demonstrated their superiority under certain conditions. Both, however, considered only conventional transformers, which (for dispersive waveguides) can further be improved, as demonstrated in this paper.

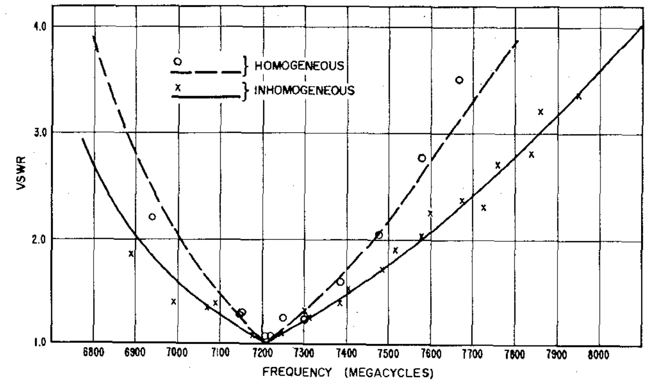

Fig. 6—Experimental points and theoretical curves for the two transformers tested.

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[14] L. Solymar, "Some notes on the optimum design of stepped transmission-line transformers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 374–378; October, 1958.

[15] H. J. Riblet, "A general theorem on an optimum stepped impedance transformer," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 169–170; March, 1960.